# Some remarks on spherical harmonics

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**Abstract**

The article contains several observations on spherical harmonics and their nodal sets: a construction for harmonics with prescribed zeroes; a natural representation of this type for harmonics on $\mathbb{S}^2$; upper and lower bounds for nodal length and inner radius (the upper bounds are sharp); the precise upper bound for the number of common zeroes of two spherical harmonics on $\mathbb{S}^2$; the mean Hausdorff measure on the intersection of $k$ nodal sets for harmonics of different degrees on $\mathbb{S}^m$, where $k \le m$ (in particular, the mean number of common zeroes of $m$ harmonics).

## Introduction

This article contains several observations on spherical harmonics and their nodal sets; the emphasis is on the case of $\mathbb{S}^2$.

Let $M$ be a compact connected homogeneous Riemannian manifold, $G$ be a compact Lie group acting on $M$ transitively by isometries, and $\mathcal{E}$ be a $G$-invariant subspace of the (real) eigenspace for some non-zero eigenvalue of the Laplace–Beltrami operator. We show that each function in $\mathcal{E}$ can be realized as the determinant of a matrix, whose entries are values of the reproducing kernel for $\mathcal{E}$. There is a similar well-known construction for the orthogonal polynomials. However, the method does not work for an arbitrary finite dimensional $G$-invariant subspace of $C(M)$ (see Remark 2). There is a natural unique up to scaling factors realization of this type for spherical harmonics on $\mathbb{S}^2$. It can be obtained by complexification and restriction to the null-cone $x^2 + y^2 + z^2 = 0$ in $\mathbb{C}^3$. There is a two-sheeted equivariant covering of this cone by $\mathbb{C}^2$, which identifies the space $\mathcal{H}_n$ of harmonic homogeneous complex-valued polynomials of degree $n$ on $\mathbb{R}^3$ with the space $\mathcal{P}^2_{2n}$ of homogeneous holomorphic polynomials on $\mathbb{C}^2$ of degree $2n$. [1]

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[1] In 1876, Sylvester used an equivalent construction to refine Maxwell's method for representation of spherical harmonics. According to it, one has to differentiate the function $1/r$, where $r$ is the distance to origin, in suitable directions in $\mathbb{R}^3$ to get a real harmonic. The directions are uniquely defined; the corresponding points in $\mathbb{S}^2$ are called poles (see [15, Ch. 9] or [3, 11.5.2]; [7, Ch. 7, section 5] and [1, Appendix A] contain extended expositions and further information).

The set of all zeroes of a real spherical harmonic $u$ is called a *nodal set.* We say that $u$ and its nodal set $N_u$ are *regular* if zero is not a critical value of $u$. Then each component of $N_u$ is a Jordan contour. According to [11], a pair of the nodal sets $N_u, N_v$, where $u, v \in \mathcal{H}_n^{\mathbb{R}}$ and $n > 0$, have a non-void intersection; moreover, if $u$ is regular, then each component of $N_u$ contains at least two points of $N_v$. The set $N_u \cap N_v$ may be infinite but the family of such pairs $(u, v)$ is closed and nowhere dense in $\mathcal{H}_n^{\mathbb{R}} \times \mathcal{H}_n^{\mathbb{R}}$. If $N_u \cap N_v$ is finite, then $\operatorname{card} N_u \cap N_v \le 2n^2$. The estimate follows from the Bezout theorem and is precise. This gives an upper bound for the number of critical points of a generic spherical harmonic, which probably is not sharp. The configuration of critical points is always degenerate in some sense (see Remark 5). The problem of finding lower bounds seems to be more difficult. According to partial results and computer experiments, $2n$ may be the sharp lower bound.

The investigation of metric and topological properties of the nodal sets has a long and rich history; we only give a few remarks on the subject of this paper. Let $\Delta$ be the Laplace–Beltrami operator and $\lambda$ be an eigenvalue of $-\Delta$.

In 1978, Brüning ([5]) found the lower bound $c\sqrt{\lambda}$ for the length of a nodal set on a Riemann surface. Yau conjectured ([22, Problem 74]) that the Hausdorff measure of the nodal set of a $\lambda$-eigenfunction on a compact Riemannian manifold admits upper and lower bounds of the type $c\sqrt{\lambda}$. This conjecture was proved by Donnelly and Fefferman for real analytic manifolds in [8]. In ([18]), Savo proved that $\frac{1}{11}\operatorname{Area}(M)\sqrt{\lambda}$ is the lower bound for the length of a nodal set in a surface $M$ for all sufficiently large $\lambda$ in any surface and for all $\lambda$ if the curvature is nonnegative. The upper and lower estimates of the inner radius were found by Mangoubi ([13], [14]); in the case of surfaces, they are of order $\lambda^{-\frac{1}{2}}$ ([13]).

One can find the 1-dimensional Hausdorff measure of a set in $\mathbb{S}^2$ integrating over $\mathrm{SO}(3)$ the counting function for the number of its common points with translates of a suitable subset of $\mathbb{S}^2$ (see Theorem 4). Using estimates of the number of common zeroes, we give upper and lower bounds for the length of a nodal set and for the inner radius of a nodal domain in $\mathbb{S}^2$. The upper bounds are precise.

Let $\mathcal{H}_n^{m+1}$ be the space of all real spherical harmonics of degree $n$ on the unit sphere $\mathbb{S}^m$ in $\mathbb{R}^{m+1}$. Corresponding to a point of $\mathbb{S}^m$ the evaluation functional at it on $\mathcal{H}_n^{m+1}$, we get an equivariant immersion of $\mathbb{S}^m$ to the unit sphere in $\mathcal{H}_n^{m+1}$, which is locally a metric homothety with the coefficient $\sqrt{\frac{\lambda_n}{m}}$, where $\lambda_n = n(n+m-1)$ is the eigenvalue of $-\Delta$ in $\mathcal{H}_n^{m+1}$. This makes it possible to calculate the mean Hausdorff measure of the intersection of $k$ harmonics of degrees $n_1, \ldots, n_k$: it is equal to $c\sqrt{\lambda_{n_1} \ldots \lambda_{n_k}}$, where $c$ depends only on $m$ and $k$ and $k \le m$ (Theorem 6). In particular, if $k = m$, then we get the mean number of common zeroes of $m$ harmonics: it is equal to $2m^{-\frac{m}{2}}\sqrt{\lambda_{n_1} \ldots \lambda_{n_m}}$; if $m = 2$, then $\sqrt{\lambda_{n_1}\lambda_{n_2}}$. In article [8], Donelly and Fefferman wrote: "A main theme of this paper is that a solution of $\Delta F = -\lambda F$, on a real analytic manifold, behaves like a polynomial of degree $c\sqrt{\lambda}$". Following this idea, L. Polterovich conjectured that the mean number of common zeroes is subject to the Bezout theorem, i.e., that it is as above. Thus, the result in the case $k = m$ confirms

this conjecture up to multiplication by a constant, and may be treated as "the Bezout theorem in the mean" for the spherical harmonics. For $k = 1$, the mean Hausdorff measure, by different but similar methods, was found by Berard in [4] and Neuheisel in [16]. The case of a flat torus was investigated by Rudnick and Wigman ([17]).

# 1 Construction of eigenfunctions which vanish on prescribed finite sets

In this section, $M$ is a compact connected oriented homogeneous Riemannian manifold of a compact Lie group $G$ acting by isometries on $M$, $\Delta$ is the Laplace–Beltrami operator on $M$,

$$\lambda > 0 \tag{1}$$

is an eigenvalue of $-\Delta$, $\mathcal{E}_\lambda$ is the corresponding real eigenspace (i.e., $\mathcal{E}_\lambda$ consists of real valued eigenfunctions), and $\mathcal{E}$ is its $G$-invariant linear subspace. Thus, $\mathcal{E}$ is a finite sum of $G$-invariant irreducible subspaces of $C^\infty(M)$. The invariant measure with the total mass 1 on $M$ is denoted by $\sigma$, $L^2(M) = L^2(M, \sigma)$. For any $a \in M$, there exists the unique $\phi_a \in \mathcal{E}$ that realizes the evaluation functional at $a$:

$$\langle u, \phi_a \rangle = u(a)$$

for all $u \in \mathcal{E}$. Set

$$\phi(a, b) = \phi_a(b), \quad a, b \in M.$$

It follows that

$$\phi(a, b) = \phi_a(b) = \langle \phi_a, \phi_b \rangle = \langle \phi_b, \phi_a \rangle = \phi_b(a) = \phi(b, a), \tag{2}$$

$$u(x) = \langle u, \phi_x \rangle = \int \phi(x, y) u(y)\, d\sigma(y) \quad \text{for all } u \in \mathcal{E}, \tag{3}$$

$$x \in N_u \iff \phi_x \perp u, \tag{4}$$

$$\phi_x \neq 0 \quad \text{for all} \quad x \in M. \tag{5}$$

The latter holds due to the homogeneity of $M$. According to (3), $\phi(x, y)$ is the *reproducing kernel for* $\mathcal{E}$ (i.e., the mapping $u(x) \to \int \phi(x, y) u(y)\, d\sigma(y)$ is the orthogonal projection onto $\mathcal{E}$ in $L^2(M)$).

Let $a_1, \dots, a_k, x, y \in M$. Set $a = (a_1, \dots, a_k) \in M^k$ and let $a$ also denote the corresponding $k$-subset of $M$: $a = \{a_1, \dots, a_k\}$. Set

$$\Phi_k^a(x, y) = \Phi_{k,y}^a(x) = \det \begin{pmatrix} \phi(a_1, a_1) & \dots & \phi(a_1, a_k) & \phi(a_1, y) \\ \vdots & \ddots & \vdots & \vdots \\ \phi(a_k, a_1) & \dots & \phi(a_k, a_k) & \phi(a_k, y) \\ \phi(x, a_1) & \dots & \phi(x, a_k) & \phi(x, y) \end{pmatrix}. \tag{6}$$

Obviously, $\Phi^a_k(x,y) = \Phi^a_k(y,x)$. Let us fix $y$ and set $v = \Phi^a_{k,y}$. Then, by (6), $v \in \mathcal{E}$ and

$$a_1, \dots a_k \in N_v. \tag{7}$$

We say that $a_1, \dots a_k$ are *independent* if the vectors $\phi_{a_1}, \dots, \phi_{a_k} \in \mathcal{E}$ are linearly independent. For a subset $X \subseteq M$, put

$$\mathcal{N}_X = \operatorname{span}\{\phi_x : x \in X\}. \tag{8}$$

If $X = N_u$, where $u \in \mathcal{E}$, then we abbreviate the notation: $\mathcal{N}_{N_u} = \mathcal{N}_u$. Set

$$n = \dim \mathcal{E} - 1.$$

It follows from (1) that $n \geq 1$ (note that $\mathcal{E}$ is real and $G$-invariant).

**Lemma 1.** *Let $a \in M^k$, where $k \leq n$. Then $a_1, \dots a_k$ are independent if and only if $\Phi^a_{k,y} \neq 0$ for some $y \in M$.*

*Proof.* It follows from (4) that $\mathcal{E} = \mathcal{N}_M$; since $k \leq n$, $\mathcal{N}_a \neq \mathcal{E}$. Therefore, if $a_1, \dots a_k$ are independent, then we get an independent set adding $y$ to $a$, for some $y \in M$. Then $\Phi^a_{k,y} \neq 0$ since $\Phi^a_{k,y}(y) > 0$ (by (2) and (6), $\Phi^a_{k,y}(y)$ is the determinant of the Gram matrix for the vectors $\phi_{a_1}, \dots, \phi_{a_k}, \phi_y$). Clearly, $\Phi^a_{k,y} = 0$ for all $y \in M$ if $a_1, \dots a_k$ are dependent. $\square$

The following proposition implies that each function in $\mathcal{E}$ can be realized in the form (6).

**Proposition 1.** *For any $u \in \mathcal{E}$, $\mathcal{N}_u = u^\perp \cap \mathcal{E}$.*

**Lemma 2.** *If $u, v \in \mathcal{E}$ and $N_v \supseteq N_u$, then $v = cu$ for some $c \in \mathbb{R}$.*

*Proof.* This immediately follows from the inclusion $N_v \supseteq N_u$ and Lemma 1 of [11], which states that $v = cu$ for some $c \in \mathbb{R}$ if there exist nodal domains $U, V$ for $u, v$, respectively, such that $V \subseteq U$. $\square$

Here is a sketch of the proof of the mentioned lemma; it is based on the same idea as Courant's Nodal Domain Theorem. Since $u$ does not change its sign in $U$, $\lambda$ is the first Dirichlet eigenvalue for $U$. Hence, it has multiplicity 1 and $D(w) \geq \lambda \|w\|_{L^2(U)}$ for all $w \in C^2(M)$ that vanish on $\partial U$, where $D$ is the Dirichlet form on $U$. Moreover, the equality holds if and only if $w = cu$ for some $c \in \mathbb{R}$. On the other hand, if $w$ vanishes outside $V$ and coincides with $v$ in $V$, then the equality is fulfilled.

*Proof of Proposition 1.* If $v \in \mathcal{E}$ and $v \perp \mathcal{N}_u$, then $N_v \supseteq N_u$ by (4). Thus, $v \in \mathbb{R}u$ by Lemma 2. Therefore, $\mathcal{N}_u \supseteq u^\perp \cap \mathcal{E}$. The reverse inclusion is evident. $\square$

Let $\Phi : M^{n+1} \to \mathcal{E}$ be the mapping $(a, y) \to \Phi^a_{n,y}$ and set $\mathcal{U} = \Phi(M^{n+1})$.

**Theorem 1.** (i) *Let $u \in \mathcal{E}$, $u \neq 0$. For $(a, y) \in N_u^n \times M$,*

$$\Phi(a, y) = c(a, y)u, \tag{9}$$

*where $c$ is a continuous nontrivial function on $N_u^n \times M$.*

(ii) *$\mathcal{U}$ is a compact symmetric neighbourhood of zero in $\mathcal{E}$.*

(iii) *For every $a \in M^n$, there exists a nontrivial nodal set which contains $a$; for a generic $a$, this set is unique.*

*Proof.* Let $a \in N_u^n$. If $a_1, \dots, a_n$ are independent, then $\operatorname{codim} \mathcal{N}_a = 1$; since $u \perp \mathcal{N}_u$ by (4), we get (9), where $c(a, y) \neq 0$ for some $y \in M$ by Lemma 1. If $a_1, \dots, a_n$ are dependent, then $\Phi(a, y) = 0$ for all $y \in M$ by the same lemma. The function $c$ is continuous by (6); it is nonzero since the set $N_u$ contains independent points $a_1, \dots, a_n$ by Proposition 1. This proves (i).

According to (6), $\Phi$ is continuous. Hence, $\mathcal{U}$ is compact. Since $M$ is connected, for any $u \in \mathcal{U}$, we may get the segment $[0, u]$ moving $y$; hence, $\mathcal{U}$ is starlike. Since transposition of every two points in $a$ changes the sign of $c(a, y)$, $\mathcal{U}$ is symmetric if $n > 1$; for $n = 1$, $\mathcal{U}$ is a disc in $\mathcal{E}$ because it is $G$-invariant and starlike. Thus, $\mathcal{U}$ is compact, symmetric, starlike, and $\cup_{t>0} t\mathcal{U} = \mathcal{E}$. Hence $\mathcal{U}$ is a neighbourhood of zero, i.e., (ii) is true.

Let $a \in M^n$ and $a' \subseteq a$ be a maximal independent subset of $a$. Then $\Phi_{k,y}^{a'} \neq 0$ for some $y \in M$ by Lemma 1, where $k = \operatorname{card} a'$. Set $v = \Phi_{k,y}^{a'}$. According to (7), $a' \subset N_v$. By (4), $N_v$ contains any point $x \in M$ such that $\phi_x \in \mathcal{N}_{a'}$. Hence $N_v$ includes $a$. The set $N_v$ is unique if $a_1, \dots, a_n$ are independent because $\operatorname{codim} \mathcal{N}_v = 1$ in this case. Since $M$ is homogeneous and $\mathcal{E}$ is finite dimensional, the functions $\phi_x$, $x \in M$, are real analytic. Therefore, either $\Phi_{n,y}^a = 0$ for all $(a, y) \in M^{n+1}$ or $\Phi_{n,y}^a \neq 0$ for generic $(a, y)$ (note that $M$ is connected). Finally, $\Phi_{n,y}^a \neq 0$ for some $(a, y) \in M^{n+1}$ since $\mathcal{N}_M = \mathcal{E}$ due to (4) and (5). $\square$

A closed subset $X \subseteq M$ is called an *interpolation set for a function space* $\mathcal{F} \subseteq C(M)$ if $\mathcal{F}|_X = C(X)$.

**Corollary 1.** *Let $k \leq \dim \mathcal{E}$. For generic $a_1, \dots, a_k \in M$, $a = \{a_1, \dots, a_k\}$ is an interpolation set for $\mathcal{E}$.* $\square$

**Remark 1.** The function $c$ may vanish on some components of the set $N_u^n \times M$. For example, let $M$ be the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ and $\mathcal{E}$ be the restriction to it of the space of harmonic homogeneous polynomials of degree $k$; then $\dim \mathcal{E} = 2k + 1$, $n = 2k$. If $k > 1$, then any big circle $\mathbb{S}^1$ in $\mathbb{S}^2$ is contained in several nodal sets (for example, nodal sets of the functions $x_1 f(x_2, x_3)$, where $f$ is harmonic, contain the big circle $\{x_1 = 0\} \cap \mathbb{S}^2$); moreover, if $k$ is odd, then $\mathbb{S}^1$ may be a component of $N_u$. Hence, $\operatorname{codim} \mathcal{N}_{\mathbb{S}^1} > 1$ and $\Phi(a, y) = 0$ for all $(a, y) \in \left(\mathbb{S}^1\right)^n \times \mathbb{S}^2$.

**Remark 2.** Theorem 1 fails for a generic finite dimensional $G$-invariant subspace $\mathcal{E} \subseteq C(M)$. Indeed, if $\dim \mathcal{E} > 1$ and $\mathcal{E}$ contains constant functions, then it

includes an open subset consisting of functions without zeroes, which evidently cannot be realized in the form (6). Furthermore, it follows from the theorem that the products $\phi_{a_1} \wedge \cdots \wedge \phi_{a_n}$ fill a neighbourhood of zero in the $n$th exterior power of $\mathcal{E}$, which may be identified with $\mathcal{E}$. This property evidently imply the interpolation property of Corollary 1 but the converse is not true; an example is the space of all homogeneous polynomials of degree $m > 1$ on $\mathbb{R}^3$, restricted to $\mathbb{S}^2$ (or the space of all polynomials of degree less than $n$ on $[0,1] \subset \mathbb{R}$, where $n > 2$).

## 2 Spherical harmonics on $\mathbb{S}^2$

Let $\mathcal{P}_n^m$ denote the space of al homogeneous holomorphic polynomials of degree $n$ on $\mathbb{C}^m$ or/and the space of all complex valued homogeneous polynomials of degree $n$ on $\mathbb{R}^m$; clearly, there is one-to-one correspondence between these spaces. Its subspace of polynomials which are harmonic on $\mathbb{R}^m$ is denoted by $\mathcal{H}_n^m$; we omit the index $m$ in $\mathcal{H}_n^m$ if $m = 3$. Then $\dim \mathcal{H}_n = 2n + 1$. The polynomials in $\mathcal{H}_n^m$, as well as their traces on the unit sphere $\mathbb{S}^{m-1} \subset \mathbb{R}^m$, are called *spherical harmonics*. They are eigenfunctions of the Laplace–Beltrami operator; if $m = 3$, then the eigenvalue is $-n(n+1)$. For a proof of these facts, see, for example, [19]. We say that $u \in \mathcal{P}_n^m$ is *real* if it takes real values on $\mathbb{R}^m$.

The standard inner product in $\mathbb{R}^m$ and its bilinear extension to $\mathbb{C}^m$ will be denoted by $\langle\ ,\ \rangle$,

$$r(v) = |v| = \sqrt{\langle v, v\rangle}, \quad v \in \mathbb{R}^m,$$

$r^2$ is a holomorphic quadratic form on $\mathbb{C}^m$. For $a \in \mathbb{C}^m$, set

$$l_a(v) = \langle a, v\rangle\,.$$

The functions $\Phi_k^a(x,y)$ admit holomorphic extensions on all variables (except for $k$). If $M = \mathbb{S}^2 \subset \mathbb{R}^3$, then the extension to $\mathbb{C}^3$ and subsequent restriction to the null-cone

$$S_0 = \{z \in \mathbb{C}^3 : r^2(z) = 0\}$$

makes it possible to construct a kind of a natural representation in the form (6), which is unique up to multiplication by a complex number, for any complex valued spherical harmonic. The projection of $S_0$ to $\mathbb{CP}^2$ is Riemann sphere $\mathbb{CP}^1$. The cone $S_0$ admits a natural parametrization:

$$\kappa(\zeta_1, \zeta_2) = (z_1, z_2, z_3) = (2\zeta_1\zeta_2, \zeta_1^2 - \zeta_2^2, i(\zeta_1^2 + \zeta_2^2)), \quad \zeta_1, \zeta_2 \in \mathbb{C}. \tag{10}$$

**Lemma 3.** *The mapping $R : \mathcal{H}_n \to \mathcal{P}_{2n}^2$ defined by*

$$Rp = p \circ \kappa$$

*is one-to-one and intertwines the natural representations of* SO(3) *and* SU(2) *in $\mathcal{H}_n$ and $\mathcal{P}_{2n}^2$, respectively.*

*Proof.* Clearly, $p \circ \kappa$ is a homogeneous polynomial on $\mathbb{C}^2$ of degree $2n$ for any $p \in \mathcal{P}_n^3$. Further, $\kappa$ is equivariant with respect to the natural actions of SU(2) in $\mathbb{C}^2$ and SO(3) in $\mathbb{C}^3$: an easy calculation with (10) shows that the change of variables $\zeta_1 \to a\zeta_1 + b\zeta_2$, $\zeta_2 \to -\bar{b}\zeta_1 + \bar{a}\zeta_2$, where $|a|^2 + |b|^2 = 1$, induces a linear transformation in $\mathbb{C}^3$ which preserves $r^2$ and leaves $\mathbb{R}^3$ invariant (in other words, the transformation of $\mathcal{P}_2^2$, induced by this change of variables, in the base $2\zeta_1\zeta_2$, $\zeta_1^2 - \zeta_2^2$, $i(\zeta_1^2 + \zeta_2^2)$ corresponds to a matrix in SO(3)). Hence $R$ is an intertwining operator. It is well known that

$$\mathcal{P}_n^3 = \mathcal{H}_n \oplus r^2 \mathcal{P}_{n-2}^3$$

(see, for example, [19]). Since $R \neq 0$ and $Rr^2 = 0$, we get $R\mathcal{H}_n \neq 0$. It remains to note that the natural representations of these groups in $\mathcal{H}_n$, $\mathcal{P}_n^2$ are irreducible. $\square$

**Corollary 2.** *For any $p \in \mathcal{H}_n \setminus \{0\}$, the set $p^{-1}(0) \cap S_0$ is the union of $2n$ complex lines; some of them may coincide. If the lines are distinct, $q \in \mathcal{H}_n$, and $p^{-1}(0) \cap S_0 = q^{-1}(0) \cap S_0$, then $q = cp$ for some $c \in \mathbb{C}$.*

*Proof.* Clearly, $\kappa$ maps lines onto lines and induces an embedding of $\mathbb{CP}^1$ into $\mathbb{CP}^2$. $\square$

The functions $\phi_a$ of the previous section can be written explicitly:

$$\phi_a(x) = c_n P_n(\langle a, x \rangle), \quad \text{where} \quad a, x \in \mathbb{S}^2,$$

$c_n$ is a normalizing constant, and $P_n$ is the $n$th Legendre polynomial: $P_n(t) = \frac{1}{2^n n!} \frac{d^n}{dt^n} (t^2 - 1)^{2n}$. There is the unique extension of

$$\phi(a, x) = \phi_a(x)$$

into $\mathbb{R}^3$ which is homogeneous of degree $n$ and harmonic on both variables (it is also symmetric and extends into $\mathbb{C}^3$ holomorphically). For example, if $n = 3$, then $2P_3(t) = 5t^3 - 3t$ and $\phi(a, x)$ is proportional to

$$5 \langle a, x \rangle^3 - 3 \langle a, a \rangle \langle a, x \rangle \langle x, x \rangle$$

(if $a = (1, 0, 0)$, then to $2x_1^3 - 3x_1x_2^2 - 3x_1x_3^2$). Of course, the representation of $p \in \mathcal{H}_n$ in the form (6) holds for $M = \mathbb{S}^2$ but there is a more natural version in this case. For $\zeta = (\zeta_1, \zeta_2) \in \mathbb{C}^2$, set

$$j\zeta = (-\zeta_2, \zeta_1).$$

**Theorem 2.** *Let $p \in \mathcal{H}_n$. Suppose that $p^{-1}(0) \cap S_0$ is the union of distinct lines $\mathbb{C}a_1, \ldots, \mathbb{C}a_{2n}$. Then there exists a constant $c \neq 0$ such that*

$$p(x)p(y) = c \det \begin{pmatrix} \langle a_1, a_1 \rangle^n & \ldots & \langle a_1, a_{2n} \rangle^n & \langle a_1, y \rangle^n \\ \vdots & \ddots & \vdots & \vdots \\ \langle a_{2n}, a_1 \rangle^n & \ldots & \langle a_{2n}, a_{2n} \rangle^n & \langle a_{2n}, y \rangle^n \\ \langle x, a_1 \rangle^n & \ldots & \langle x, a_{2n} \rangle^n & \langle x, y \rangle^n \end{pmatrix} \tag{11}$$

*for all $y \in S_0$, $x \in \mathbb{C}^3$. Moreover, replacing $\langle x, y\rangle^n$ with $\phi(x,y)$ in the matrix, we get such a representation of $p(x)p(y)$ for all $x, y \in \mathbb{C}^3$ (with another $c$ in general).*

*Proof.* A calculation shows that $\langle a, x\rangle^n$ is harmonic on $x$ for all $n$ if $a \in S_0$. Hence, the function $\Phi^a_y(x) = \Phi^a(x, y)$ in the right-hand side belongs to $\mathcal{H}_n$ for each $y \in S_0$. Clearly, $\Phi^a_y(a_k) = 0$ for all $k = 1, \dots, 2n$. By Corollary 2, $\Phi^a_y$ is proportional to $p$. Since $\Phi^a(x, y) = \Phi^a(y, x)$, we get (11) if the right-hand side is nontrivial. Thus, we have to prove that $c \neq 0$. Let $x \in S_0$. There exist $\alpha_1, \dots, \alpha_{2n}, \xi, \eta \in \mathbb{C}^2$ such that $a_k = \kappa(\alpha_k)$ for all $k$, $x = \kappa(\xi)$, and $y = \kappa(\eta)$. By a straightforward calculation, for any $a, b \in \mathbb{C}^2$ we get

$$\langle \kappa(a), \kappa(b)\rangle = -2\langle a, jb\rangle^2 . \tag{12}$$

Hence, the right-hand side of (11) is equal to

$$-2^{(2n+1)n} c \det \begin{pmatrix} \langle \alpha_1, j\alpha_1\rangle^{2n} & \dots & \langle \alpha_1, j\alpha_{2n}\rangle^{2n} & \langle \alpha_1, j\eta\rangle^{2n} \\ \vdots & \ddots & \vdots & \vdots \\ \langle \alpha_{2n}, j\alpha_1\rangle^{2n} & \dots & \langle \alpha_{2n}, j\alpha_{2n}\rangle^{2n} & \langle \alpha_{2n}, j\eta\rangle^{2n} \\ \langle \xi, j\alpha_1\rangle^{2n} & \dots & \langle \xi, j\alpha_{2n}\rangle^{2n} & \langle \xi, j\eta\rangle^{2n} \end{pmatrix}. \tag{13}$$

The determinant can be calculated explicitly. More generally, if $C = (c_{rs})_{r,s=1}^{k+1}$, where $c_{rs} = \langle a_r, b_s\rangle^k$, $a_r, b_s \in \mathbb{C}^2$, then

$$\det C = \prod_{r=1}^{k} \binom{k}{r} \prod_{s<r} \langle a_r, ja_s\rangle \prod_{s<r} \langle b_r, jb_s\rangle \tag{14}$$

Let $a_r = (a_{r,1}, a_{r,2})$, $b_s = (b_{s,1}, b_{s,2})$. If all the entries are nonzero, then

$$c_{rs} = \sum_{t=0}^{k} \binom{k}{r} (a_{r,1}b_{s,1})^t (a_{r,2}b_{s,2})^{k-t} = a^k_{r,2} b^k_{s,1} \sum_{t=0}^{k} \binom{k}{r} \left(\frac{a_{r,1}}{a_{r,2}}\right)^t \left(\frac{b_{s,2}}{b_{s,1}}\right)^{k-t}$$

We may factor out rows and columns of $C$. Then we get a matrix $\tilde{C}$, which admits the decomposition $\tilde{C} = AB$, where

$$A = \left(\tbinom{k}{r}\alpha_r^t\right)_{r,t=0}^{k}, \quad B = \left(\beta_s^{k-t}\right)_{t,s=0}^{k}, \quad \alpha_r = \tfrac{a_{r,1}}{a_{r,2}}, \quad \beta_s = \tfrac{b_{s,2}}{b_{s,1}}.$$

Thus, the computation of $\det C$ is reduced to the Vandermonde determinant. The straightforward calculation proves (14); obviously, the assumption that the entries are nonzero is not essential. Due to (14) and (12), this implies that the determinant in (13) is not zero if the lines $\mathbb{C}\xi, \mathbb{C}\eta, \mathbb{C}a_1, \dots, \mathbb{C}a_{2n}$ are distinct (if $a \in S_0$, then the plane $\langle z, a\rangle = 0$ intersects $S_0$ in the line $\mathbb{C}a$). Hence, $c \neq 0$.

It follows from the definition of $P_n$ and $\phi$ that

$$\phi(x, y) = s_n \langle x, y\rangle^n + r^2(x) r^2(y) h(x, y), \tag{15}$$

there $s_n > 0$ is constant and $h$ is a polynomial. Therefore, we can get a function $f \neq 0$ on $\mathbb{C}^3$, which coincides with $p(x)$ on $S_0$ up to a constant factor, replacing $\langle x, y\rangle^n$ with $\phi(x, y)$ in (11) and fixing generic $y \in \mathbb{C}^3$. By Corollary 2, the same is true on $\mathbb{C}^3$ since $f \in \mathcal{H}_n$ according to (11) (all functions in the last row are harmonic on $x$). Since $\phi(x, y) = \phi(y, x)$, this proves the second assertion. $\square$

**Remark 3.** The set $p^{-1}(0) \cap S_0$, where $p \in \mathcal{H}_n$, is also distinguished by the orthogonality condition

$$\int_{\mathbb{S}^2} p(x) \langle x, w\rangle^n \, d\sigma(x) = 0,$$

where $\sigma$ is the invariant measure on $\mathbb{S}^2$ and $w \in S_0$. This is a consequence of (15) since $\int p(x)\phi(x, y)\, d\sigma(x) = p(y)$ for all $y \in \mathbb{S}^2$, hence for all $y \in \mathbb{R}^3$ ($p(y)$ and $\phi_x(y)$ are homogeneous of degree $n$), moreover, for all $y \in \mathbb{C}^3$ (both sides are holomorphic on $y$). In particular, this is true for $y \in S_0$ but $\phi(x, y) = s_n \langle x, y\rangle^n$ in this case.

If $p^{-1}(0) \cap S_0$ is the union of distinct lines $\mathbb{C}a_k$, $k = 1, \dots, 2n$, then the functions $\langle x, a_k\rangle^n$, $k = 1, \dots, 2n$, form a linear base for the space of functions in $\mathcal{H}_n$ which are orthogonal to $p$ with respect to the bilinear form $\int fg\, d\sigma$. This is a consequence of (12): it is easy to check that the functions $\langle \zeta, b_s\rangle^k$ on $\mathbb{C}^2$, where $s = 1, \dots, k$, are linearly independent if the lines $\mathbb{C}b_s$ are distinct (the Vandermonde determinant). $\square$

We conclude this section with remarks on number of zeroes in $\mathbb{S}^2$ of functions in $\mathcal{H}_n$. Let $f \in \mathcal{H}_n$, $u = \operatorname{Re} f$, $v = \operatorname{Im} f$. A zero of $f$ is a common zero of $u$ and $v$. The following proposition, in a slightly more general form, was proved in [11]. We say that $u$ is *regular* if zero is not a critical value for $u$.

**Proposition 2** ([11])**.** *Let $n > 0$, $u \in \mathcal{H}_n$. If $u$ is regular, then for any $v \in \mathcal{H}_n$ each connected component of $N_u$ contains at least two points of $N_v$.* $\square$

The assertion follows from the Green formula which implies that

$$\int_C v \frac{\partial u}{\partial n}\, ds = 0, \tag{16}$$

where $C$ is a component of $N_u$, which is a Jordan contour, $ds$ is the length measure on $C$, and $\frac{\partial u}{\partial n}$ is the normal derivative; note that $\frac{\partial u}{\partial n}$ keeps its sign on $C$. For the standard sphere $\mathbb{S}^2$, (16) follows from the classical Green formula for the domain $D_\varepsilon = (1 - \varepsilon, 1 + \varepsilon) \times \mathbb{S}^2$, where $\varepsilon \in (0, 1)$, and the homogeneous of degree 0 extensions of $u, v$ into $D_\varepsilon$.

Let $u, v \in \mathcal{H}_n$ be real and regular. Set

$$\nu(u, v) = \operatorname{card} N_u \cap N_v.$$

For singular $u, v$, zeroes must be counted with multiplicities; if $u, v \in \mathcal{H}_n$, then the multiplicity of a zero can be defined as the number of smooth nodal lines which meet at it; if $u, v$ have multiplicities $k, l$ at their common zero, then one

have to count them $kl$ times (the greatest number of common zeroes which appear under small perturbations). If $u = \phi_a$, where $a \in \mathbb{S}^2$, then $N_u$ is the union of $n$ parallel circles $\langle x, a\rangle = t_k$, $x \in \mathbb{S}^2$, where $k = 1, \dots, n$ and $t_1, \dots, t_n$ are the zeroes of $P_n(t)$. Since they are distinct, $P_n'(t_k) \neq 0$ for all $k$. It follows from Proposition 2 that for any real $v \in \mathcal{H}_n$

$$\nu(\phi_a, v) \geq 2n,$$

where $a \in \mathbb{S}^2$. If $b \in \mathbb{S}^2$ is sufficiently close to $a$, then the equality holds for $v = \phi_b$. In the inequality above, $\phi_a$ and $n$ may be replaced with any regular $u$ and the number of components of $N_u$, respectively. The latter can be less than $n$ (according to [12], it can be equal to one or two if $n$ is odd or even, respectively[2]). However, computer experiments support the following conjecture: for all real $u, v \in \mathcal{H}_n$,

$$\nu(u, v) \geq 2n.$$

The common zeroes must be counted with multiplicities. Otherwise, there is a simple example of two harmonics which have only two common zeroes: $\mathrm{Re}(x_1 + ix_2)^n$ and $\mathrm{Im}(x_1 + ix_2)^n$.

On the other hand, for generic real $u, v \in \mathcal{H}_n$ there is a trivial sharp upper bound for $\nu(u, v)$. We prove a version that is stronger a bit.

**Proposition 3.** *Let $u, v \in \mathcal{H}_n$ be real. If $\nu(u, v)$ is finite, then*

$$\nu(u, v) \leq 2n^2. \tag{17}$$

By the Bezout theorem, if $u, v \in \mathcal{P}_n^3$ have no proper common divisor, then the set $\{z \in \mathbb{C}^3 : u(z) = v(z) = 0\}$ is the union of $n^2$ (with multiplicities) complex lines. Then $\nu(u, v) \leq 2n^2$ since each line has at most two common points with $\mathbb{S}^2$. The proposition is not an immediate consequence of this fact since $u, v$ may have a nontrivial common divisor which has a finite number of zeroes in $\mathbb{S}^2$. This cannot happen for $u, v \in \mathcal{H}_n$ by the following lemma.

**Lemma 4.** *Let $u \in \mathcal{H}_n$ be real, $x \in \mathbb{S}^2$, and $u(x) = 0$. Suppose that $u = vw$, where $v \in \mathcal{P}_m^3$, $w \in \mathcal{P}_{n-m}^3$ are real. If $w(y) \neq 0$ for all $y \in \mathbb{S}^2 \setminus \{x\}$ that are sufficiently close to $x$, then $w(x) \neq 0$.*

*Proof.* We may assume $x = (0, 0, 1)$. If $u$ has a zero of multiplicity $k$ at $x$, then

$$u(x_1, x_2, x_3) = p_k(x_1, x_2)x_3^{n-k} + p_{k+1}(x_1, x_2)x_3^{n-k-1} + \dots + p_n(x_1, x_2),$$

where $p_j \in \mathcal{P}_j^2$, $p_k \neq 0$. Since $\Delta u = 0$, we have $\Delta p_k = 0$. Hence,

$$p_k(x_1, x_2) = \mathrm{Re}(\lambda(x_1 + ix_2)^k)$$

[2]The corresponding harmonic is a small perturbation of the function $\mathrm{Re}(x_1 + ix_2)^n$.

for some $\lambda \in \mathbb{C} \setminus \{0\}$. Therefore, $p_k$ is the product of $k$ distinct linear forms. Let

$$w = q_l(x_1, x_2)x_3^{n-m-l} + q_{l+1}(x_1, x_2)x_3^{n-m-l-1} + \cdots + q_{n-m}(x_1, x_2),$$
$$v = r_s(x_1, x_2)x_3^{m-s} + r_{s+1}(x_1, x_2)x_3^{m-s-1} + \cdots + r_m(x_1, x_2),$$

where $q_j, r_j \in \mathcal{P}_j^2$ and $q_l, r_s \neq 0$. Since $p_k = q_l r_s$, we have $k = l + s$; moreover, either $q_l$ is constant or it is the product of distinct linear forms. The latter implies that it change its sign near $x$; then the same is true for $w$, contradictory to the assumption. Hence $l = 0$. Thus, $q_l \neq 0$ implies $w(x) = q_l(x) \neq 0$. $\square$

*Proof of Proposition 3.* Let $u, v \in \mathcal{H}_n$ be real and $w$ be their greatest common divisor. Clearly, $w$ is real. Since $N_u \cap N_v$ is finite, zeroes of $w$ in $\mathbb{S}^2$ must be isolated; by Lemma 4, $w$ has no zero in $\mathbb{S}^2$. Applying the Bezout theorem to $u/w$ and $v/w$, we get the assertion. $\square$

The equality in (17) holds, for example, for the following pairs and for their small perturbations:

$$u = \phi_a, \quad v = \operatorname{Re}(x_2 + ix_3)^n, \quad \text{where} \quad a = (1, 0, 0); \tag{18}$$
$$u = \operatorname{Re}(ix_2 + x_3)^n, \quad v = \operatorname{Re}(x_1 + ix_2)^n.$$

**Corollary 3.** *If the number of critical points for real $u \in \mathcal{H}_n$ is finite, then it does not exceed $2n^2$; in particular, this is true for a generic real $u \in \mathcal{H}_n$.*

*Proof.* If $x$ is a critical point of $u$, then $\xi u(x) = 0$ for any vector field $\xi \in \mathrm{so}(3)$. It is possible to choose two fields $\xi, \eta \in \mathrm{so}(3)$ which do not annihilate $u$ and are independent at all critical points; then the critical points of $u$ are precisely the common zeroes of $\xi u, \eta u \in \mathcal{H}_n$. $\square$

**Remark 4.** This bound is not sharp. At least, for $n = 1, 2$ the number of critical points is equal to $2(n^2 - n + 1)$, if it is finite. Let $u, v$ be as in (18). Then $u + \varepsilon v$, where $\varepsilon$ is small, has $2(n^2 - n + 1)$ critical points. I know no example of a spherical harmonic with a greater (finite) number of critical points.

**Remark 5.** The consideration above proves a bit more than Corollary 3 says. A nontrivial orbit of $u$ under SO(3) is either 3-dimensional or 2-dimensional, and the latter holds if and only if $u = c\phi_a$ for some constant $c$ and $a \in \mathbb{S}^2$. In the first case, the set $C$ of critical points of $u$ is precisely the set of common zeroes of three linearly independent spherical harmonics (a base for the tangent space to the orbit of $u$). Hence, $\operatorname{codim} \mathcal{N}_C \geq 3$. Note that generic three harmonics have no common zero. Thus, the configuration of critical points is always degenerate. The problem of estimation of the number of critical points, components of nodal sets, nodal domains, etc., for spherical harmonics on $\mathbb{S}^2$ was stated in [2].

**Proposition 4.** *The set $\mathcal{I}$ of functions $f = u + iv \in \mathcal{H}_n$ such that $\nu(u, v) = \infty$ is closed and nowhere dense in $\mathcal{H}_n$.*

*Proof.* If $N_u \cap N_v$ is infinite, then it contains a Jordan arc which extends to a contour since $u$ and $v$ are real analytic (by [6], a nodal set, outside of its finite subset, is the finite union of smooth arcs). This contour cannot be included into a disc $D$ which is contained in some of nodal domains: otherwise, its first Dirichlet eigenvalue would be greater than $n(n+1)$. Therefore, diameter of the contour is bounded from below. This implies that $\mathcal{I}$ is closed. If $f \in \mathcal{I}$, then $u$ and $v$ have a nontrivial common divisor due to the Bezout theorem; hence, $\mathcal{I}$ is nowhere dense. $\square$

In examples known to me, if $f \in \mathcal{I}$, then $N_u \cap N_v$ is the union of circles.

## 3 Estimates of nodal length and inner radius

Let $M$ be a $C^\infty$-smooth compact connected Riemannian manifold, $m = \dim M$, $\mathfrak{h}^k$ be the $k$-dimensional Hausdorff measure on $M$. Yau conjectured that there exists positive constant $c$ and $C$ such that

$$c\sqrt{\lambda} \leq \mathfrak{h}^{m-1}(N_u) \leq C\sqrt{\lambda}$$

for the nodal set $N_u$ of any eigenfunction $u$ corresponding to the eigenvalue $-\lambda$. For real analytic $M$, this conjecture was proved by Donnelly and Fefferman in [8]. In the case of a surface, lower bounds were obtained in papers [5] and [18]; in [18], $c = \frac{1}{11}\,\mathrm{Area}(M)$.

We consider first the case $M = \mathbb{S}^m \subset \mathbb{R}^{m+1}$, $m \geq 1$. Set

$$\psi(x) = \mathrm{Re}(x_1 + ix_2)^n.$$

Clearly, $\psi \in \mathcal{H}_n^{m+1}$. Let $\phi$ denote a zonal spherical harmonic; we omit the index since the geometric quantities that characterize $N_\phi$ are independent of it. Set

$$\omega_k = \mathfrak{h}^k(\mathbb{S}^k) = \frac{2\pi^{\frac{k+1}{2}}}{\Gamma\left(\frac{k+1}{2}\right)}.$$

**Theorem 3.** *For any nonzero real $u \in \mathcal{H}_n^{m+1}$,*

$$\mathfrak{h}^{m-1}(N_u) \leq \mathfrak{h}^{m-1}(N_\psi) = n\omega_{m-1}. \tag{19}$$

The theorem is simply an observation modulo the following fact (a particular case of Theorem 3.2.48 in [10]). A set which can be realized as the image of a bounded subset of $\mathbb{R}^k$ under a Lipschitz mapping is called *$k$-rectifiable* (we consider only the sets which can be realized as the countable union of compact sets). Since $u \in \mathcal{H}_n^{m+1}$ is a polynomial, the set $N_u$ is $(m-1)$-rectifiable. Let $\mu_m$ denote the invariant measure on $\mathrm{O}(m+1)$ with the total mass 1.

**Theorem 4** ([10])**.** *Let $A, B \subseteq \mathbb{S}^d$ be compact, $A$ be $k$-rectifiable, and $B$ be $l$-rectifiable. Set $r = k + l - d$. Suppose $r \geq 0$. Then*

$$\int_{\mathrm{O}(d)} \mathfrak{h}^r(A \cap gB)\, d\mu_d(g) = K\mathfrak{h}^k(A)\mathfrak{h}^l(B), \tag{20}$$

*where* $K = \dfrac{\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{l+1}{2}\right)}{2\Gamma\left(\frac{1}{2}\right)^d\Gamma\left(\frac{r+1}{2}\right)} = \dfrac{\omega_r}{\omega_k\omega_l}$. □

If $r = 0$, then the left-hand side of (20) is a version of the Favard measure for spheres (on $A$ or $B$). Also, note that (20) can be proved directly in this setting since the left-hand side, for fixed $A$ (or $B$), is additive on finite families of disjoint compact sets; thus, it is sufficient to check its asymptotic behavior on small pieces of submanifolds.

**Lemma 5.** *For any real* $u \in \mathcal{H}_n^{m+1}$ *and each big circle* $\mathbb{S}^1$ *in* $\mathbb{S}^m$*, if* $\mathbb{S}^1 \cap N_u$ *is finite, then*

$$\operatorname{card}(\mathbb{S}^1 \cap N_u) \leq 2n. \tag{21}$$

*Proof.* The restriction of $u$ to the linear span of $\mathbb{S}^1$, which is 2-dimensional, is a homogeneous polynomial of degree $n$ of two variables. □

*Proof of Theorem 3.* Since $\mathbb{S}^1$ intersects in two points any hyperplane which does not contain it, for almost all $g \in \mathrm{O}(m+1)$ we have

$$\operatorname{card}(g\mathbb{S}^1 \cap N_u) \leq 2n = \operatorname{card}(g\mathbb{S}^1 \cap N_\psi).$$

Integrating over $\mathrm{O}(m+1)$ and applying (20) with $k = 1$, $l = m-1$, $A = \mathbb{S}^1$, $B = N_u$ and $B = N_\psi$, we get the inequality in (19). The equality is evident. □

A lower bound can also be obtained in this way. In what follows, we assume $k = l = 1$ and $m = 2$; then $K = \frac{1}{2\pi^2}$, and (19) read as follows:

$$\mathfrak{h}^1(N_u) \leq 2\pi n. \tag{22}$$

The nodal set $N_\phi$ of a zonal spherical harmonic $\phi = \phi_a \in \mathcal{H}_n$, where $a \in \mathbb{S}^2$, is the union of parallel circles of Euclidean radii $\sqrt{1-t_k^2}$, where $t_k$ are zeroes of the Legendre polynomial $P_n$. The smallest circle corresponds to the greatest zero $t_n$. Set $r_n = \sqrt{1-t_n^2}$ and let $C_n$ be a circle in $\mathbb{S}^2$ of Euclidean radius $r_n$. By Proposition 2, for any $u \in \mathcal{H}_n$,

$$\operatorname{card}(gC_n \cap N_u) \geq 2 \quad \text{for all} \quad g \in \mathrm{O}(3). \tag{23}$$

Due to (20),

$$\mathfrak{h}^1(N_u) \geq \frac{2\pi}{r_n}\,.$$

By [21, Theorem 6.3.4], $t_n = \cos\theta_n$, where

$$0 < \theta_n < \frac{j_0}{n+\frac{1}{2}} \tag{24}$$

and $j_0 \approx 2.4048$ is the least positive zero of Bessel function $J_0$. This estimate, by [21, (6.3.15)], is asymptotically sharp: $\lim_{n\to\infty} n\theta_n = j_0$. Thus,

$$r_n = \sin\theta_n < \sin\frac{j_0}{n+\frac{1}{2}} < \frac{j_0}{n+\frac{1}{2}},$$

and we get

$$\mathfrak{h}^1(N_u) > \frac{2\pi}{j_0}\left(n+\frac{1}{2}\right). \tag{25}$$

The bound (25) is not the best one but it is greater than $\frac{1}{11}\,\mathrm{Area}\,(M)\sqrt{\lambda}$, the bound of paper [18]:

$$\frac{4\pi}{11}\sqrt{n(n+1)} < \frac{2\pi}{j_0}\left(n+\frac{1}{2}\right),$$

since $\frac{4\pi}{11} \approx 1.4248$, $\frac{2\pi}{j_0} \approx 2.6127$; according to [18], $\frac{1}{11}\,\mathrm{Area}\,(M)\sqrt{\lambda}$ estimates from below the nodal length for all closed Riemannian surfaces $M$ (for sufficiently large $\lambda$ in general and for all $\lambda$ if the curvature is nonnegative). The length of the nodal set of a zonal harmonic could be the sharp lower bound. According to [21, (6.21.5)], $\frac{k-\frac{1}{2}}{n+\frac{1}{2}}\pi \le \tau_{n-k} \le \frac{k}{n+\frac{1}{2}}\pi$, where $\cos\tau_k$, $k=0,\dots,n-1$, are the zeroes of $P_n$ in the order of decreasing (i.e., $\tau_1 = \theta_n$). Hence

$$\mathfrak{h}^1(N_\phi) = 2\pi\sum_{k=1}^{n}\sin\theta_k \approx 2\pi n\int_0^1 \sin\pi x\,dx = 4n$$

as $n\to\infty$. If this is true, then the upper bound is rather close to the lower one since their ratio tends to $\frac{\pi}{2}$ as $n\to\infty$.

It is also possible to estimate the *inner radius* of $\mathbb{S}^2 \setminus N_u$:

$$\mathrm{inr}(\mathbb{S}^2\setminus N_u) = \sup\left\{\inf_{y\in N_u}\rho(x,y) : \, x\in\mathbb{S}^2\right\},$$

where $\rho$ is the inner metric in $\mathbb{S}^2$:

$$\rho(x,y) = \arccos\langle x,y\rangle\,.$$

The least upper bound is evident:

$$\mathrm{inr}(\mathbb{S}^2\setminus N_u) \le \mathrm{inr}(\mathbb{S}^2\setminus N_\phi) = \theta_n$$

by (24). Indeed, it is attained for $u=\phi$ and cannot be greater since the circle $C_n$ intersects any nodal set by Proposition 2. Let $C(\theta)$ be the a circle of radii $\theta$ in the inner metric of $\mathbb{S}^2$; then Euclidean radius of $C(\theta)$ is $r=\sin\theta$. A number $\theta_0 > 0$ is a lower bound for the inner radius if and only if the following conditions hold:

(i) $\theta_0 \le \theta_n$,

(ii) for each real $u \in \mathcal{H}_n$, there exists $g \in \mathrm{O}(3)$ such that $gC(\theta_0) \cap N_u = \emptyset$.

(note that the disc bounded by $C(\theta_0)$ cannot contain a component of $N_u$ due to (i)). Further, for almost all $g \in \mathrm{O}(3)$ the number $\mathrm{card}(gC(\theta_0) \cap N_u)$ is even. Therefore, we may assume that

$$\mathrm{card}(gC(\theta_0) \cap N_u) \geq 2$$

if $gC(\theta_0) \cap N_u \neq \emptyset$. Set $r_0 = \sin\theta_0$. If (ii) is false then

$$2 \leq \frac{1}{2\pi^2}\mathfrak{h}^1(C(\theta_0))\mathfrak{h}^1(N_u) = \frac{r_0}{\pi}\,\mathfrak{h}^1(N_u) \leq 2r_0 n$$

by (20). Thus, if $r_0 < \frac{1}{n}$, then $\theta_0$ is a lower bound for $\mathrm{inr}(\mathbb{S}^2 \setminus N_u)$. Hence $\arcsin\frac{1}{n}$ is a lower bound for $\mathrm{inr}(\mathbb{S}^2 \setminus N_u)$. The estimate seems to be non-sharp; perhaps, the least inner radius has the set $\mathbb{S}^2 \setminus N_\psi$ (it is equal to $\frac{\pi}{2n}$).

We summarize the results on $\mathbb{S}^2$.

**Theorem 5.** *Let $M = \mathbb{S}^2$. For any nonzero real $u \in \mathcal{H}_n$,*

$$\tfrac{2\pi}{j_0}\left(n + \tfrac{1}{2}\right) < \mathfrak{h}^1(N_u) \leq 2\pi n, \tag{26}$$

$$\arcsin\tfrac{1}{n} \leq \mathrm{inr}\left(\mathbb{S}^2 \setminus N_u\right) \leq \theta_n < \tfrac{j_0}{n+\frac{1}{2}}. \tag{27}$$

*In (26), the upper bound is attained if $u = \psi$; the upper bound $\theta_n$ in (27) is attained for $u = \phi$.* □

# 4 Mean Hausdorff measure of intersections of the nodal sets

Let us fix $m \geq 2$ and the unit sphere $\mathbb{S}^m \subset \mathbb{R}^{m+1}$. We shall find the mean value over $u_1, \dots, u_k$, $k \leq m$, of the Hausdorff measure of sets

$$N_{u_1} \cap \cdots \cap N_{u_k} \subset \mathbb{S}^m.$$

If $k = m$, then this is the mean number of common zeroes of $u_1, \dots, u_m$ in $\mathbb{S}^m$. Set

$$\mathbf{n} = (n_1, \dots, n_k),$$
$$\delta(n) = \dim \mathcal{H}_n^{m+1} - 1,$$

where $n, n_j$ are positive integers. We define the mean value as follows:

$$M_{\mathbf{n}} = \int_{\mathbb{S}^{\delta(n_1)} \times \cdots \times \mathbb{S}^{\delta(n_k)}} \mathfrak{h}^{m-k}\left(N_{u_1} \cap \cdots \cap N_{u_k}\right) d\tilde{\sigma}_{\delta(n_1)}(u_1) \dots d\tilde{\sigma}_{\delta(n_k)}(u_k), \tag{28}$$

where $\tilde{\sigma}_j$ denotes the invariant measure on $\mathbb{S}^j$ with the total mass 1. Let $\lambda_n$ be the eigenvalue of $-\Delta$ in $\mathcal{H}_n^{m+1}$; recall that

$$\lambda_n = n(n + m - 1).$$

**Theorem 6.** *Let $1 \le k \le m$. Then*

$$M_{\mathbf{n}} = \omega_{m-k} m^{-\frac{k}{2}} \sqrt{\lambda_{n_1} \dots \lambda_{n_k}}, \tag{29}$$

*where $M_{\mathbf{n}}$ is defined by* (28).

If $k = m$, then we get the mean value of card $(N_{u_1} \cap \dots \cap N_{u_m})$; since $\omega_0 = 2$ and $\mathfrak{h}^0 =$ card, it is equal to

$$2m^{-\frac{m}{2}} \sqrt{\lambda_{n_1} \dots \lambda_{n_m}}.$$

There is a natural equivariant immersion $\iota_n : \mathbb{S}^m \to \mathbb{S}^{\delta(n)} \subset \mathcal{H}_n^{m+1}$:

$$\iota_n(a) = \frac{\phi_a}{|\phi_a|}. \tag{30}$$

If $n$ is odd, then $\iota_n$ is one-to-one; for even $n > 0$, $\iota_n$ is a two-sheeted covering, which identifies opposite points. Clearly, the Riemannian metric in $\iota(\mathbb{S}^m)$ is $\mathrm{O}(m+1)$-invariant and the stable subgroup of $a$ acts transitively on spheres in $T_a\mathbb{S}^m$. Hence, the mapping $\iota_n$ is locally a metric homothety. Let $s_n$ be its coefficient. Clearly,

$$s_n = \frac{|d_a \iota_n(v)|}{|v|}, \tag{31}$$

where the right-hand side is independent of $a \in \mathbb{S}^m$ and $v \in T_a\mathbb{S}^m \setminus \{0\}$. For any $l$-rectifiable set $X \subseteq \mathbb{S}^m$ such that $X \cap (-X) = \emptyset$, where $l \le m$, we have

$$\mathfrak{h}^l(\iota_n(X)) = s_n^l \mathfrak{h}^l(X). \tag{32}$$

**Lemma 6.** *Let $u \in \mathcal{H}_n^{m+1}$ and $X \subseteq \mathbb{S}^m$ be compact, symmetric, and $(r+1)$-rectifiable, where $r \le m-1$. Then*

$$\int_{\mathbb{S}^{\delta(n)}} \mathfrak{h}^r(N_u \cap X)\, d\sigma_{\delta(n)}(u) = s_n \frac{\omega_r}{\omega_{r+1}} \mathfrak{h}^{r+1}(X).$$

*Proof.* Since both sides are additive on $X$, we may assume $X \cap (-X) = \emptyset$. We apply Theorem 4 to the sphere $\mathbb{S}^{\delta(n)}$ and its subsets $A = \mathbb{S}^{\delta(n)-1}$, $B = \iota_n(X)$. In the notation of this theorem, $d = \delta(n)$, $k = d-1$, $l = r+1$; $K\omega_k = \frac{\omega_r}{\omega_l}$. Replacing integration over $\mathbb{S}^d$ by averaging over $\mathrm{O}(d+1)$ and using (32), we get

$$\begin{aligned} \int_{\mathbb{S}^d} \mathfrak{h}^r(N_u \cap X)\, d\sigma_d(u) &= \frac{1}{s_n^r} \int_{\mathbb{S}^d} \mathfrak{h}^r(\iota(N_u \cap X))\, d\sigma_d(u) \\ = \frac{1}{s_n^r} \int_{\mathbb{S}^d} \mathfrak{h}^r(u^\perp \cap \iota(X))\, d\sigma_d(u) &= \frac{1}{s_n^r} \int_{\mathrm{O}(d+1)} \mathfrak{h}^r(g\mathbb{S}^k \cap \iota(X))\, d\mu_d(g) \\ = \frac{1}{s_n^r} K \mathfrak{h}^k\left(\mathbb{S}^k\right) \mathfrak{h}^{r+1}(\iota(X)) = \frac{\omega_r}{s_n^r \omega_{r+1}} \mathfrak{h}^{r+1}(\iota(X)) &= s_n \frac{\omega_r}{\omega_{r+1}} \mathfrak{h}^{r+1}(X). \end{aligned}$$

$\square$

**Corollary 4.** *The mean value of* $\mathfrak{h}^{m-1}(N_u)$ *over* $u \in \mathcal{H}_n^{m+1}$ *is equal to* $s_n\omega_{m-1}$.

*Proof.* Set $X = \mathbb{S}^m$, $r = m-1$. ☐

**Corollary 5.** *Let* $M_{\mathbf{n}}$*,* $m$*, and* $k$ *be as in* (28)*. Then*

$$M_{\mathbf{n}} = \omega_{m-k} \prod_{j=1}^{k} s_{n_j}. \tag{33}$$

*Proof.* Set $X = N_{u_1} \cap \cdots \cap N_{u_{k-1}}$. By Lemma 6,

$$M_{\mathbf{n}} = s_{n_k} \frac{\omega_{m-k}}{\omega_{m-k+1}} M_{\mathbf{n}'},$$

where $\mathbf{n}' = (n_1, \ldots, n_{k-1})$. Applying this procedure repeatedly and using Corollary 4 in the final step, we get (33). ☐

It remains to find $s_n$. Set

$$d = \dim \mathrm{O}(m+1).$$

Since the stable subgroup $\mathrm{O}(m)$ of the point $a = (0, \ldots, 0, 1)$ acts transitively on spheres in $T_a\mathbb{S}^m$, the invariant Riemannian metric in $\mathbb{S}^m$ can be lifted up to a bi-invariant metric on $\mathrm{O}(m+1)$ in such a way that the canonical projection $\mathrm{O}(m+1) \to \mathbb{S}^m$ is a metric submersion. Let $\xi_1, \ldots, \xi_m, \ldots, \xi_d$ be an orthonormal linear base in the Lie algebra $\mathrm{so}(m+1)$. Realizing $\mathrm{so}(m+1)$ by the left invariant vector fields on $\mathrm{O}(m+1)$, we get the invariant Laplace–Beltrami operator on $\mathrm{O}(m+1)$:

$$\tilde{\Delta} = \xi_1^2 + \cdots + \xi_d^2.$$

The sum is independent of the choice of the base since it is left invariant and this property holds at the identity element $e$. Thus, we may assume that

$$\xi_{m+1}, \ldots, \xi_d \in \mathrm{so}(m). \tag{34}$$

For $f \in C^2(\mathbb{S}^m)$, set $\tilde{f}(g) = f(ga)$. Then $\langle \Delta f, \phi_a \rangle = \tilde{\Delta}\tilde{f}(e)$. Since $\iota$ is equivariant, we have

$$d_a\iota(\xi a) = \frac{1}{|\phi_a|}\xi\phi_a \tag{35}$$

for all $\xi \in \mathrm{so}(m+1)$. It follows from (34) that $\xi_1 a, \ldots, \xi_m a$ is a base for $T_a\mathbb{S}^m$ and $\xi_1\phi_a, \ldots \xi_m\phi_a$ is a base for $T_{\phi_a}\iota(\mathbb{S}^m)$. Moreover,

$$|\xi_k a| = 1, \quad k = 1, \ldots, m,$$
$$\xi_k a = 0, \quad k = m+1, \ldots, d,$$

where the first equality holds since the projection $\mathrm{O}(m+1) \to \mathbb{S}^m$ is a metric submersion. Due to these equalities, (30), (31), and (35), we get

$$ms_n^2 = s_n^2 \sum_{k=1}^{d} |\xi_k a|^2 = \sum_{k=1}^{d} |d_a \iota(\xi_k a)|^2 = \frac{1}{|\phi_a|^2} \sum_{k=1}^{d} |\xi_k \phi_a|^2$$
$$= -\frac{1}{|\phi_a|^2} \sum_{k=1}^{d} \left\langle \xi_k^2 \phi_a, \phi_a \right\rangle = -\frac{1}{|\phi_a|^2} \left\langle \Delta\phi_a, \phi_a \right\rangle = \lambda_n.$$

*Proof of Theorem 6.* Due to the calculation above,

$$s_n = \sqrt{\frac{\lambda_n}{m}}.$$

Thus, Corollary 5 implies (29). □

In the case $n_1 = \cdots = n_k = n$, there is another natural explanation of the equalities (29), (33):

$$M_{\mathbf{n}} = \omega_{m-k} \left( \frac{\lambda_n}{m} \right)^{\frac{k}{2}} = \omega_{m-k} s_n^k.$$

The mean value can be defined as the average over the action of the group $\mathrm{O}(m+1)$ on the set of subspaces of codimension $k$ in $\mathcal{H}_n^{m+1}$, which can be realized as $\mathcal{N}_{u_1} \cap \cdots \cap \mathcal{N}_{u_k} = u_1^{\perp} \cap \cdots \cap u_k^{\perp}$:

$$\begin{aligned} M_{\mathbf{n}} &= \int_{\mathrm{O}(m+1)} \mathfrak{h}^{m-k}(\iota_n^{-1}(g\mathbb{S}^{\delta(n)-k} \cap \iota_n(\mathbb{S}^m)))\, d\mu_m(g) \\ &= s_n^{k-m} \int_{\mathrm{O}(m+1)} \mathfrak{h}^{m-k}(g\mathbb{S}^{\delta(n)-k} \cap \iota_n(\mathbb{S}^m))\, d\mu_m(g) \\ &= s_n^{k-m} \tfrac{\omega_{m-k}}{\omega_m} \mathfrak{h}^m(\iota(\mathbb{S}^m)) \quad = \omega_{m-k} s_n^k. \end{aligned}$$

The method of calculation of the mean Hausdorff measure easily can be extended to families of invariant (may be, reducible) finite dimensional function spaces on a homogeneous space whose isotropy group acts transitively on spheres in the tangent space.

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